

Sporadic randomness, Maxwell's Demon and the Poincaré recurrence times

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In the case of fully chaotic systems the distribution of the Poincaré recurrence times is an exponential whose decay rate is the Kolmogorov-Sinai (KS) entropy. We address the discussion of the same problem, the connection between dynamics and thermodynamics, in the case of sporadic randomness, using the Manneville map as a prototype of this class of processes. We explore the possibility of relating the distribution of Poincaré recurrence times to “thermodynamics”, in the sense of the KS entropy, also in the case of an inverse power law. This is the dynamic property that Zaslavsky (G.M. Zaslavsky, *Physics Today* **52**(8), 39 (1999)) finds to be responsible for a striking deviation from ordinary statistical mechanics under the form of Maxwell's Demon effect.

We show that this way of establishing a connection between thermodynamics and dynamics is valid only in the case of strong chaos, where both the sensitivity to initial conditions and the distribution of the Poincaré recurrence times are exponential. In the case of sporadic randomness, resulting at long times in the Lévy diffusion processes, the sensitivity to initial conditions is initially a power law, but it becomes exponential again in the long-time scale, whereas the distribution of Poincaré recurrence times keeps, or gets, its inverse power law nature forever, including the long-time scale where the sensitivity to initial condition becomes exponential.

We show that a non-extensive version of thermodynamics would imply the Maxwell's Demon effect to be determined by memory, and thus to be temporary, in conflict with the dynamic approach to Lévy statistics. The adoption of heuristic arguments indicate that this effect is possible, as a form of genuine equilibrium, after completion of the process of memory erasure.

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I. INTRODUCTION

In a lucid discussion Lebowitz [1] has recently restated the point of view of Boltzmann to establish the microscopic origin of irreversible macroscopic behavior. In his view the adoption of the laws of big numbers is essential, and the role of deterministic chaos becomes important only if it applies to a macroscopic number of non-interacting particles. According to Lebowitz mixing and ergodicity are notions “unnecessary, misguided and misleading”. In other words, this opinion reflects the conviction, mirrored by the handbooks of statistical mechanics, that the unification of mechanics and thermodynamics rests on $N \rightarrow \infty$, where N denotes the number of degrees of freedom of the system under study.

This point of view has to be contrasted with a new view about the origin of thermodynamics [2] produced by the increasing interest in nonlinear dynamics. Within this new perspective the sensitivity to the initial conditions, rather than the large number of degrees of freedom involved, is playing the crucial role of generator of thermodynamics. Zaslavsky [3] has recently discussed the phenomenon of the Poincaré recurrences without using the traditional argument that these recurrences occur in a so extended time scale as to lie beyond the observation range. Using the Bernoulli shift map, as a dynamic

prototype of strong chaos, Zaslavsky proves [3] that the distribution of Poincaré recurrence times, $P_R(t)$, obeys the following prescription:

$$P_R(t) \propto \exp(-h_{KS}t), \quad (1)$$

where h_{KS} denotes the Kolmogorov-Sinai (KS) entropy [2]. Zaslavsky makes the interesting remark that Eq.(1) establishes a nice connection between mechanics, the left hand side of this equation, and thermodynamics, the right hand side of the same equation. This is so because h_{KS} has an entropic significance [2]. It is also well known that the KS entropy is related to the Lyapunov coefficient and thus to the sensitivity to initial conditions, as a consequence of the Pesin theorem [4]. To make the connection between $P_R(t)$ and the Lyapunov coefficient perspective more transparent, it is convenient to introduce also the property:

$$\xi(t) \equiv \lim_{\Delta x_0 \rightarrow 0} \left| \frac{\Delta x(t)}{\Delta x(0)} \right|, \quad (2)$$

where $\Delta x(t)$ denotes the difference between the space coordinates of two trajectories, moving from two distinct initial conditions at a distance $|\Delta x(0)|$ the one from the other. It is evident that after identifying h_{KS} with the Lyapunov coefficient λ , we can rewrite Eq.(1) in the form

$$P_R(t) \propto \xi(-t) = \frac{1}{\xi(t)}, \quad (3)$$

where

$$\xi(t) = \exp(\lambda t). \quad (4)$$

At a first sight, this point of view seems to conflict with that of Lebowitz. In this paper we shall argue, with a heuristic approach, that it is not so, and that a sort of compromise between the two views can be found. Here we want to limit ourselves to noticing that the observation frequently made that the occurrence of strong chaos is exceptional, does not necessarily mean that thermodynamics can only rest on the $(N \rightarrow \infty)$ -perspective. It might only imply that the process of transition to equilibrium and the ensuing statistics are not ordinary. The same aspect might emerge also from within the adoption of the $(N \rightarrow \infty)$ -perspective. We have two examples in mind, where the thermodynamic level, if it exists, is closely related to anomalous rather than ordinary statistical mechanics. The first is the dynamical process recently discussed by Latora *et al.* [5]. This is a Hamiltonian system of N rotors, each rotor being coupled to all the other rotors of the system with the same coupling. The authors show that in the limit $N \rightarrow \infty$ from this Hamiltonian picture a process of superdiffusion under the form of a Lévy walk emerges. On the other hand, we have in mind a second example, of Hamiltonian nature, resulting in Lévy diffusion. This example is afforded by the class of billiards discussed in the recent book of Zaslavsky [3]. These are systems with only one degree of freedom, thereby seemingly departing from the condition $N \rightarrow \infty$ that, according to Lebowitz [1], is essential for the emergence of thermodynamics. In this paper we shall try to prove that the point of view advocated by Lebowitz [1] is compatible with that sustained by Zaslavsky [3,6,7] in the case of fully chaotic systems, where the dynamical approach to statistical mechanics yields the same ordinary form of statistical mechanics as the $(N \rightarrow \infty)$ -perspective, thereby making apparently useless the adoption of a dynamic view. However, we shall argue that the role of dynamics might become important when the system under study is characterized by persistent memory. We make also the conjecture that a kind of statistical equivalence exists between extended memory and long-range interactions.

For the time being, we would like to stress that in his very interesting book [3], as well as in the original papers [6,7], Zaslavsky focuses his attention on the anomalous thermodynamic nature of these generators of anomalous diffusion. He shows that two chaotic billiards, coupled to one another through a small hole in the wall separating one billiard from the other, result in a so strong violation of the condition of equal distribution as to suggest the occurrence of a Maxwell's Demon effect. Maxwell's Demon was described by Maxwell in his book "Theory

of Heat" (1871) and represented a rather simple device. Two chambers, A and B , are separated by a division with a small hole. Positioned at the hole the demon allows swifter molecules to pass from chamber A to B and slower molecules to pass from B to A . After a while chamber A will contain mainly swifter molecules, whereas chamber B will contain mainly slower molecules. This implies the appearance of temperature difference without expenditure of work and contradicts the Second Law of Thermodynamics. The version of Maxwell's Demon realized in practice in Refs. [3,6,8,7] is equivalent to the case where particles with the same kinetic energy tends to spend more time in one of the two chambers even if the volume available is the same.

According to Zaslavsky this surprising behavior is provoked by the fact that the distribution density $P_R(t)$ deviates from the thermodynamic condition of Eq.(1). This is so because even in the phase space of seemingly chaotic systems there might exist stable islands. The surface of separation between the chaotic sea and the stable islands is characterized by fractal properties and these properties make these surfaces very sticky. Consequently, in the long-time limit the function $P_R(t)$ becomes proportional to $\psi(t)$, denoting the distribution of waiting times at the border between chaotic sea and stability island. Using renormalization group arguments it is shown [3] that

$$\lim_{t \rightarrow \infty} \psi(t) = \frac{\text{const}}{t^\mu}, \quad (5)$$

where $\mu > 2$, in accordance with the Kac theorem [9]. This means that the first moment of the waiting time distribution is finite. After revealing the sources of deviations from the ordinary prescriptions of thermodynamics and statistical mechanics, Zaslavsky does not rule out the possibility that even in this case a thermodynamic perspective applies. He states [8] that all this is obliging us to rethink the foundation of thermodynamics.

We see that there are several issues to discuss:

(i) Is there a connection between the condition of strong chaos leading to thermodynamics in the sense of Eq. (1) and the Boltzmann condition $N \rightarrow \infty$? Is there, in general, a connection between the perspective based on deterministic chaos of low-dimensional systems and the perspective based on $N \rightarrow \infty$?

(ii) As far as the connection between $\psi(t)$ and the sensitivity to initial condition is concerned, we see that in the case where the exponential picture does not apply Eq.(3) splits into the following two different possibilities:

$$\psi(t) \propto \xi(-t) \quad (6)$$

and

$$\psi(t) \propto \frac{1}{\xi(t)}. \quad (7)$$

Which is the correct property?

(iii) Is there a form of thermodynamics involved in this case? If it is so, which is the nature of this form of thermodynamics? This would answer the fundamental question raised by Zaslavsky about the origin of his Maxwell's Demon effect [3,8,6].

(iv) Is it possible to find a general analytical expression for $\psi(t)$, fitting the inverse power law of Eq.(5), and collapsing, for a critical value of a given parameter, into the exponential form of Eq. (1) with the same coefficient h_{KS} ?

To give the reader a preliminary understanding of question (iv) we have to make some remarks. We shall show that the Manneville map [10] is a dynamic process general enough as to reproduce both the exponential property of Eq.(1) and the inverse power law of Eq.(5), depending on the value of a given control parameter z . We shall study this dynamic system while keeping in mind the non-extensive thermodynamics introduced by Tsallis in 1988 [11], which is now becoming more and more popular [12]. From a formal point of view, within this new form of statistical mechanics a crucial role is played by the so called q -exponential. If we express the function $\psi(t)$ as a q -exponential we obtain

$$\psi(t) = [1 - (1 - q)h_q t]^{1/(1-q)}. \quad (8)$$

The parameter q is called entropic index. We see immediately that with $q > 1$ the function $\psi(t)$ is an inverse power law. At the critical value $q = 1$ the function $\psi(t)$ would make an abrupt transition from the inverse power law Eq.(5) to the exponential form of Eq. (1). An important motivation for this paper has been to assess whether or not the non-extensive thermodynamics of Tsallis might be a satisfactory solution of question (iii) as well as of (iv).

This paper is organized as follows. In Section II we discuss the dynamical properties of the Manneville map. We shall discuss theoretically and numerically the waiting time distribution $\psi(t)$ in this specific case. In Section III we illustrate very simple arguments supporting the choice of Eq.(7). Section IV is devoted to exploring the possibility that the Tsallis non-extensive thermodynamics might shed light on the thermodynamic nature of the Maxwell's Demon effect. Section V shows that the extended regime of transition from dynamics to thermodynamics corresponds to the steady process of memory erasure accompanying the dynamic realization of Lévy processes. Finally Section VI is devoted to making a balance on the answers to questions (i)-(iv) that we are giving throughout this paper.

II. THE MANNEVILLE MAP

In this section we illustrate some key dynamic properties of the Manneville map [10]. The reader can find

surprising that after mentioning the key issues of Section I, we decide to focus our attention on a non-Hamiltonian system, with only one variable. We feel therefore the need of justifying our choice. First of all, we want to remark that, as proved by the authors of Ref. [5], the use of a Hamiltonian system with infinitely many degrees of freedom does not rule out the possibility that the resulting dynamics, as far as a few collective variables are concerned, is equivalent to a random walk process that can be obtained dynamically also from a map [13]. For the emergence of low-dimensional chaos from a Hamiltonian picture with a large number of degrees of freedom we refer to the discussion of Tennyson *et al.* [14].

Then there is another reason why the choice of the Manneville map is convenient to mimic the Hamiltonian properties responsible for the birth of the Lévy diffusion. According to Zaslavsky, the Lévy superdiffusion in one of his two-dimensional systems is generated by the fact that the states of uniform motion last with a waiting time distribution that has an inverse power law nature. This means that any condition of uniform motion corresponds to a stable island whose border with the surrounding chaotic sea is fractal. A trajectory sticks to this border with an inverse power law distribution of waiting times. For the whole sojourn time of the trajectory on this border, in the laboratory frame of reference the system undergoes a uniform motion in a given direction. From time to time the trajectory leaves the fractal region and after a short sojourn in the chaotic sea it sticks again to another fractal region. The particle sticking to a given fractal region moves in the laboratory frame of reference with constant velocity. The change from one sticking condition to another might have the effect of changing the motion direction, but it does not affect the velocity modulus. Note that the velocity modulus is constant because the trajectory motion occurs in the constant energy surface of the phase space. A nice dynamical model sharing all these properties is the egg-carton model of Geisel *et al.* [15], where the trajectory moves of uniform motion along a given channel of the egg-crate landscape and at a given time, randomly selected from an inverse power law distribution, begins moving in one of the two opposite directions of a orthogonal channel. Also this direction of motion is randomly selected. Notice that *randomly* here refers to a property of deterministic chaos.

The one-dimensional version of this generator of anomalous diffusion corresponds to a velocity ξ with only two possible values, W and $-W$. These two states must have a distribution of sojourn times with an inverse power law. Thus, we see immediately that the one-dimensional version of the Hamiltonian systems studied by Zaslavsky becomes similar to the one-dimensional map of Ref. [13]. This map has been widely used in the recent past to derive Lévy processes [16–19]. On the other hand, for the purpose of the discussion of the present paper, as already done in the earlier work of Ref. [20], it is con-

venient to shift our attention from the bimodal maps of Refs. [13,16–19] to the case of the Manneville map, which has only one laminar region. The Kolmogorov complexity of the Manneville map has been already analyzed years ago by Gaspard and Wang [21]. The complexity of a sequence equivalent to that of the bimodal maps of Refs. [13,16–19] was studied recently by Buiatti *et al.* [?]. It is straightforward to show that the Kolmogorov complexity of the Manneville map is proportional to the complexity of the bimodal maps generating Lévy processes [23]. We are interested in sporadic randomness, and for our purposes it is irrelevant whether after exiting from one laminar region and crossing the chaotic region, the trajectory is injected back into the original laminar region, or into a new laminar region equivalent to the original [23]. If we make the arbitrary decision that crossing the chaotic region always implies a transition from one laminar region to the other, the resulting complexity is the same as that of the Manneville map, with only one laminar region, and consequently no uncertainty associated with its choice. In the bimodal case, the injection back to the original laminar region and the jump into the other laminar region are random processes, with the same probability. This implies a Kolmogorov complexity which is twice as that of the Manneville map [23].

In conclusion, the reason for the choice of the Manneville is that this map mimics very well the condition of sporadic randomness that, according to Zaslavsky [3], is responsible for Maxwell’s Demon effect. Therefore, we are convinced that a numerical treatments of the Hamiltonian systems with sporadic randomness would lead to the same conclusions as those reached by us here, on the basis of the Manneville map.

The Manneville map reads:

$$x_{n+1} = \Phi(x_n) = x_n + x_n^z (\text{mod}.1) (1 \leq z). \quad (9)$$

We note that at $z = 1$ the Manneville map becomes equivalent to the Bernoulli shift map used by Zaslavsky to prove the fundamental result of Eq. (1). For $z > 1$ the interval $[0, 1]$ is divided in two regions, the laminar region, $[0, d(z)]$, and the chaotic region, $[d(z), 1]$, with $d(z)$ defined by

$$d(z) + d(z)^z = 1 \quad (10)$$

We review here the arguments used by Geisel and Thomae [13] to derive an analytical expression for the distribution of the times of sojourn in the laminar region. First of all we assume that the injection point x_0 is so close to $x = 0$ as to replace eq.(9) with:

$$dx/dt = x^z. \quad (11)$$

Thus we obtain the following time evolution:

$$x(t) = [x_0^{1-z} + (1-z)t]^{1/(1-z)}. \quad (12)$$

Hence the time necessary for the trajectory to get the border $x = d(z)$ is given by

$$t = T(x_0) \equiv \left(\frac{1}{x_0^{z-1}} - \frac{1}{d^{z-1}} \right) \frac{1}{z-1}. \quad (13)$$

Note that in the special case where the initial condition is so close to $x = 0$ as to fulfill the condition $x_0 \ll d$, the exit time $T(x_0)$ can be satisfactorily approximated by

$$T(x_0) \approx \frac{1}{x_0^{z-1}} \frac{1}{z-1}, \quad (14)$$

which, as we shall see in Section III, can be used to define the time at which we lose control of the trajectories departing from a region of the map very close to $x = 0$.

Note that the distribution function $\psi(t)$ is related to the injection probability $p(x_0)$ by

$$\psi(t)dt = p(x_0)dx_0. \quad (15)$$

Assuming equiprobability for the injection process we have:

$$p(x_0)dx_0 = \frac{1}{d(z)} \left| \frac{dx_0}{dT} \right| dT. \quad (16)$$

Thus we obtain:

$$\psi(t) = \frac{1}{d(z)} \left| \frac{dx_0}{dt} \right|. \quad (17)$$

By differentiating t with respect to x_0 we finally arrive at

$$\psi(t) = d^{z-1} [1 + d^{z-1}(z-1)t]^{-z/(z-1)}. \quad (18)$$

It is important to observe that the mean waiting time T_{av} is given by

$$T_{av} = \frac{1}{d^{z-1}} \frac{1}{2-z}, \quad (19)$$

with $T_{av} = \infty$ for $2 \leq z$

We see that Eq.(18) implies that the region corresponding to $z > 2$ is characterized by a diverging first moment. This region is in conflict with the Kac theorem [9] and for this reason we do not take it into account. The region pertaining to the interval $1.5 < z < 2$ is characterized by a finite second moment and a diverging second moment. This is the region of interest for us, since it corresponds to that generating Lévy diffusion according to the recent work of Ref. [20]. The region $1 < z < 1.5$ is characterized by a finite second moment. Of course, the ideal condition $z = 1$ implies all the moments to be finite. This is a region which would correspond, in the perspective of Ref. [20], to ordinary Gaussian diffusion. We note that in the region $1 < z < 1.5$ the waiting function distribution must make a transition from the inverse power law behavior of Eq.(18) to the exponential regime, where

the arguments of Zaslavsky leading to Eq.(1) apply. In fact, at $z = 1$, the Manneville map becomes identical to the Bernoulli shift map. In that case, the theoretical remarks of Zaslavsky yield:

$$\psi(t) \propto \exp(-t \ln 2). \quad (20)$$

We are now in a position to make a preliminary observation about question (iv) of Section I. The analytical expression of Eq.(18) can be expressed in the form of the q -exponential of Eq.(8), with

$$q = 1 + (z - 1)/z. \quad (21)$$

However, this would not fit completely the request of being an exact representation of the waiting function in the fully chaotic case. In fact $q = 1$ would correctly imply $z = 1$, but this would make Eq.(18) collapse into

$$\psi(t) \propto \exp(-t). \quad (22)$$

rather than in the exact expression of Eq.(20).

This, in principle, would not rule out the possibility of finding a different analytical expression fulfilling the requirement of question (iv). To explore more deeply issue (iv) we use a numerical treatment of the Manneville map of Eq. (9). For computational purposes we found to be more convenient to evaluate the population of the laminar region, $M(t)$, rather than the waiting time distribution $\psi(t)$. The two functions are related the one to the other by

$$M(t) = 1 - \int_0^t \psi(t') dt'. \quad (23)$$

Thus, the analytical expression of Eq.(18) yields

$$M(t) = [1 + d^{z-1}(z-1)t]^{-1/(z-1)}. \quad (24)$$

The explicit expression of $M(t)$ in the fully chaotic case ($z = 1$) is derived from Eq.(1). In fact, Zaslavsky proves [3] that Eq.(1) becomes exact in the case of the Bernoulli map, with $h_{KS} = \ln 2$. In the normalized form, Eq.(1) becomes:

$$\psi(t) = h_{KS} \exp(-h_{KS} t). \quad (25)$$

Let us plug Eq.(25) into Eq.(23). We obtain

$$M(t) = \exp(-t \ln 2). \quad (26)$$

We know that Eq.(26) is exact and we expect Eq.(24) to become exact for $z \rightarrow \infty$. In conclusion, as to the important function $\psi(t)$, we know its exact expression at $z = 1$ and in the asymptotic limit $z \rightarrow \infty$. It is worth exploring the intermediate region with a numerical treatment.

Figs. 1 and 2 illustrate the results of our numerical treatment.

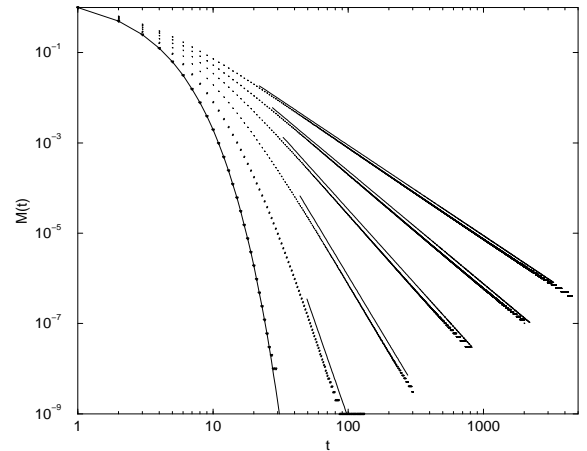


FIG. 1. $M(t)$ as a function of time. The meaning of the six full lines is as follows. The lowest full line is the function $M(t) = \exp(-t \ln 2)$. All the other full lines denote the long-time inverse power law $M(t) = 1/t^{\frac{1}{z-1}}$. The dotted lines are the numerical result. All the full lines but the lowest have been shifted to the right to make them distinguishable from the numerical result. The value of the parameter z , from the bottom to the top is: $z = 1, 1.1, 1.2, 1.3, 1.4, 1.5$

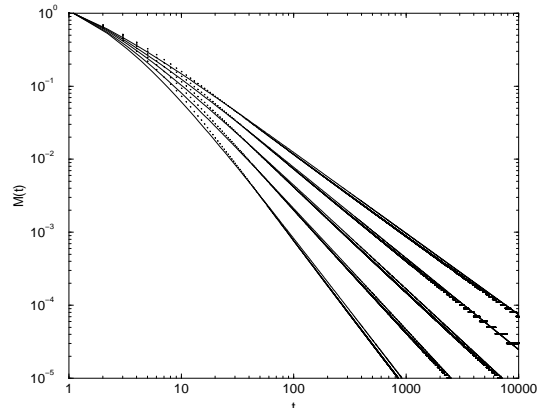


FIG. 2. $M(t)$ as a function of time. The meaning of the four pair of lines is as follows. The full lines denote the function $M(t)$ of Eq.(24) and the dotted lines denote the numerical results. To make the full lines distinguishable from the dotted lines we shifted them to the right by the quantity $\epsilon = 0.1$. In the logarithmic representation adopted, this is equivalent to replacing t of $M(t)$ with $t \exp(-\epsilon)$. The value of the parameter z from the bottom to the top changes as follows: $z = 1.5, 1.6, 1.7, 1.8, 1.9$

In Fig. 1 we illustrate the result of the numerical calculation with the parameter z in the interval $[1, 1.5]$. We see that the long-time limit fits for all values of z considered but $z = 1$, the theoretical prescription of the inverse power law $t^{-\frac{1}{z-1}}$. However, this inverse power law regime is reached after an extended transition regime, which seems to be exponential-like. The duration of this transition regime becomes more and more extended with z coming closer and closer to $z = 1$. At $z = 1$ this transition regime becomes infinitely extended and coincident with the theoretical prediction of Eq.(26).

In Fig. 2, devoted to studying the relaxation of the Manneville map with z in the interval $[1.5, 2]$, we see that the prediction of Eq.(24) is very accurate and tends to become exact with increasing the value of z .

In conclusion, these numerical results prove that the q -exponential of Eq.(8) cannot satisfactorily fit the conditions set by question (iv). However, this is not yet a strong reason to rule out the non-extensive thermodynamics of Tsallis [11,12]. Let us make another attempt at establishing this non-extensive thermodynamics as the proper form of thermodynamics that Zaslavsky is looking for. We now set a request more flexible than the one earlier adopted. This is based on disregarding the extended regime of transition to the inverse power law. In other words, we now accept an analytical form of q -exponential which at $q = 1$ becomes identical to Eq.(20) while at $z > 1$ results in the same inverse power law as the Manneville map. We try with the following possibility:

$$\psi(t) = [1 + (1 - q)h_{KS}t]^{1/(1-q)}, \quad (27)$$

supplemented by Eq.(21). To a first sight, this possibility looks very attractive. In fact, at $q = 1$, the prescription of Eq.(21) yields $z = 1$. This means the Bernoulli map, and, consequently [2]

$$h_{KS} = \ln 2. \quad (28)$$

Thus, the requirement of recovering Eq.(20) is fulfilled, while the request of producing the correct power law at $z > 1$ is obviously satisfied due to the Eq.(21).

However, this is not physically correct. We let the reader know that years ago, Gaspard and Wang [21] evaluated the Lyapunov coefficient of the Manneville map and found it to be given with a good accuracy by $h_{KS} = (2 - z)z \ln 2$. This means that h_{KS} vanishes at $z = 2$, thereby yielding a strong disagreement between Eq.(27) and Eq.(18).

One might think that the departure of Eq.(27) from Eq.(18) is caused by the fact that the coefficient h_q appearing in Eq.(8) cannot be identified with the ordinary Lyapunov coefficient. In fact, according to the Pesin theorem [4] we have

$$h_{KS} = \lambda \equiv \int_0^1 dx \rho(x) \ln(d\Phi/dx). \quad (29)$$

One might think that the adoption of Tsallis non-extensive thermodynamics implies that the ordinary definition of Lyapunov coefficient cannot be maintained and that consequently h_q cannot be identified with the property of Eq.(29), which is in fact the property discussed by Gaspard and Wang [21]. In Section IV B we shall prove that also this possibility has to be ruled out. At the same time we shall show that the correct answer to question (ii) cannot be given by Eq.(6). The derivation of the correct answer to question (ii) is given in Section III.

III. SENSITIVITY TO INITIAL CONDITIONS

The sensitivity to initial conditions of the Manneville map has been studied in Ref. [20]. The authors of this paper evaluated the function $\xi(t)$ defined by Eq.(2), using the same assumptions as those adopted in Section II to derive Eq. (18). The result is given by:

$$\xi(t) = [1 - (z - 1)x_0^{z-1}t]^{-z/(z-1)}. \quad (30)$$

This function is, in a sense, faster than the exponential, since it diverges at the time T_{div} given by

$$T_{div} \equiv \frac{1}{(z - 1)x_0^{z-1}}, \quad (31)$$

while the case of exponential sensitivity would yield finite values of $\xi(t)$ for any finite value of time.

It would be tempting to conclude that the Manneville map is characterized by a form of sensitivity to initial conditions stronger than the usual exponential sensitivity, but it would be wrong. Let us explain why it is so using heuristic arguments. Let us divide the phase space of the Manneville map into two regions. The first is very close to $x = 0$, and it is characterized by the local Lyapunov coefficients

$$\lambda(x_0) = \ln(1 + zx_0^{z-1}) \approx zx_0^{z-1}. \quad (32)$$

The second region includes the whole chaotic region plus a portion of the laminar region and it is characterized by local Lyapunov coefficients of the order of

$$\lambda \approx \ln(1 + z). \quad (33)$$

Let us now define the time at which we lose control of a trajectory, in the case of exponential sensitivity, as a time proportional to $1/\lambda$, where λ is the Lyapunov coefficient. We see that the time at which we lose control of the trajectories departing from the first region is of the order of the time scale T_{div} , of Eq.(31), while the time at which we lose control of the trajectories departing from the second region is of the order of unity. We note that for values of x_0 very close to $x = 0$ we can fulfill the inequality:

$$T_{av} \ll T_{div}, \quad (34)$$

where T_{av} is the waiting mean time defined by Eq.(19). Note also that for x_0 so small as to fit the condition of Eq.(34), the time T_{div} can be interpreted in two distinct ways. The former is the time at which $\xi(t)$ diverges, and the second is the inverse of the local Lyapunov coefficient of Eq.(32). We see, in conclusion, that the complexity of the Manneville map is decided by the time scales $1/\log(1 + z)$ and T_{av} rather than by T_{div} . This is the reason why the randomness detected by the method

of Kolmogorov complexity [21] is strongly determined by the influence of the chaotic region of the Manneville map.

This is a key point worth of a comment. The function $\xi(t)$ with the form of Eq.(30) cannot be used for the statistical interpretation of the dynamic process under study for times larger than the exit times, and consequently cannot have any relevance when the time T_{av} of Eq.(19) is finite, namely for $z < 2$. In this case, in the time scale $t \gg T_{av}$ the analytical form of the function $\xi(t)$ changes and becomes exponential [21]. At this stage there are no more possible connections between $\psi(t)$ and $\xi(t)$ left. In fact, the function $\psi(t)$ has a meaning completely different from that of $\xi(t)$. The function $\psi(t)$ signals the probability of observing sojourn times of any length, including those exceeding the time scale T_{av} . Thus, the function $\psi(t)$ maintains its inverse power law nature even in the time scale where $\xi(t)$ is exponential.

In accordance with this remarks we find a connection between the function $\psi(t)$ and the function $\xi(t)$ at the exit time T using Eq.(7). The exit time depends on the initial condition of the trajectory within the laminar region and is given by Eq. (13). Let us express x_0 of Eq.(30) in terms of this exit time, and let us replace it into Eq. (30). The resulting expression has to be interpreted as the value that the function $\xi(t)$ gets at the exit time. Its explicit expression is:

$$\xi(T) = d^z[d^{1-z} + (z-1)T]^{z/(z-1)}. \quad (35)$$

Comparing Eq.(35) to Eq.(18) we reach the conclusion that:

$$\psi(T) \propto \frac{1}{\xi(T)}. \quad (36)$$

Thus with very simple arguments we are led to conclude that the generalization of Eq.(3) implies the choice of Eq.(7). The meaning of this conclusion is that the value of $\psi(T)$ is determined by that of the function $\xi(t)$ with t being a given exit time. This very simple argument settles question (ii) of Section 1. We have to address now the much more delicate question of the connection with the thermodynamic perspective. Before ending this section, we would like to notice that the condition of Eq.(36) does not apply only in the region $z > 1.5$ which favors the continuous approximation behind the analytical results of Eqs.(30) and (18). It applies very well also to the long-time limit in the region $1 < z < 1.5$, as clearly illustrated by Fig. 1.

IV. THERMODYNAMIC APPROACH TO THE WAITING TIME DISTRIBUTION

Here we show that the entropic arguments of the non-extensive thermodynamics [12] would naturally yield Eq.(6) rather than Eq.(7), creating a conflict with the

conclusions of Section III. Thus the entropic arguments yield a wrong conclusion, which by itself does not necessarily imply that the Tsallis non-extensive thermodynamics is ill founded. If the Tsallis non-extensive thermodynamics were well founded, on the other hand, this wrong conclusion would certainly imply that this form of thermodynamics [11,12] is not permanent. In Section V we shall show, on the other hand, that the Maxwell's Demon effect is permanent and takes place in a later time scale .

A. Entropic derivation of $\psi(t)$

The dynamic derivation of Lévy processes from intermittent maps [18,19] has established that probability for the particle to make a jump of length $|x|$ in a time t , $\Pi(x)$, is related to $\psi(t)$ by

$$\Pi(x) = \psi(x/W)/W. \quad (37)$$

To make immediately evident to the reader the explanation of the form of Eq.(37) it is enough for us to let him/her know that the dynamic derivation of Lévy processes [18,19] rests on a one-dimension motion with a fluctuating velocity. The particle velocity fluctuates among the values W and $-W$. The particle sojourns in these two states with a distribution of waiting times given by $\psi(t)$, which has the same structure as that of Eq.(18). This accounts for Eq.(37). On the other hand, on the basis of the increasing interest for the subject of non-extensive entropy [12], and along lines similar to those adopted by earlier work [25–27], we decide to maximize the Tsallis entropy [11] to determine the form of $\Pi(x)$. Therefore we define

$$S_q[\Pi(x)] = -\frac{1 - \int_{-\infty}^{\infty} \Pi(x)^q dx}{(1-q)}, \quad (38)$$

and we maximize it under the constraint of assigning a fixed value to the first moment of the escort distribution with entropic index q . This means that, as done in Ref. [20], we have to maximize the functional form $F_q(\Pi)$ defined by

$$F_q(\Pi) \equiv \frac{1 - \int_{-\infty}^{\infty} \Pi(x)^q dx}{(1-q)} - \beta \int_{-\infty}^{\infty} |x| \Pi(x)^q dx + \alpha \int_{-\infty}^{\infty} \Pi(x) dx. \quad (39)$$

As a result of the entropy maximization we get [20]

$$\Pi(x) = \frac{A}{[1 + \beta(q-1)|x|^\nu]^{1/(q-1)}}. \quad (40)$$

It is evident that for this entropic argument to be compatible with the statistical derivation of $\psi(t)$ as resulting from Eq.(18) we are forced to accept the relation of

Eq.(21). In the region $1 \leq z \leq 1.5$, as shown in Fig. 1, the time regime of validity of the inverse power law fulfilling Eq.(21) is confined to regions of larger and larger times as $z \rightarrow 1$, since the size of the early time region with exponential-like behavior becomes more and more extended.

We notice that the Tsallis information approach affords a formal justification for the birth of an inverse power law. However, the special value of Eq.(21) is apparently established in such a way as to make the entropic argument compatible with the dynamic argument, with *ad hoc* arguments, rather than as a result of a theoretical prediction. The role of the entropy of Eq.(38) would become much more important if it were possible to use it with no recourse to *ad hoc* arguments. In Section IV B we shall show that perhaps it is possible to establish a dynamic approach to this non-extensive thermodynamics, and consequently to a $q > 1$ established with no *ad hoc* assumption, with the caution, however, of interpreting this thermodynamics as temporary. We shall discuss also in which sense the entropic index given by Eq.(21) as to be thought of as “magic”.

B. A transient form of Kolmogorov-Sinai entropy

The results found years ago by Gaspard and Wang [21] that the ordinary Lyapunov coefficient is finite in the range $1 \leq z \leq 2$ seem to rule out the possibility of applying to the present case the prediction of Eq.(1). In fact, since h_{KS} is finite, we can use Eq.(1), but this would lead to an exponential $\psi(t)$ in deep contrast with the fact that is an inverse power law instead. The breakdown of this important theoretical prediction is certainly due to the fact that the demonstration made by Zaslavsky [3] is incompatible with the existence of sporadic randomness.

Here we plan to discuss whether this connection between $\psi(t)$ and the KS entropy can be extended provided that a form of non-extensive KS entropy is used to examine the regions of times shorter than the mean waiting time T_{av} . To discuss this important possibility we must review the recent theoretical result of Ref. [24]. This paper provides arguments for the non-extensive generalization of the Pesin theorem [4]. As shown in Ref. [22], the proper non-extensive generalization of the Kolmogorov-Sinai entropy should rest on the calculation of

$$H_q(N) \equiv \frac{1 - \sum_{\omega_0 \dots \omega_{N-1}} p(\omega_0 \dots \omega_{N-1})^q}{q - 1}, \quad (41)$$

where $p(\omega_0 \dots \omega_{N-1})$ is the probability of finding the cylinder corresponding to the sequence of symbols $\omega_0 \dots \omega_{N-1}$ [2]. In the case where we set $q = 1$ the entropy of Eq.(41) becomes the ordinary Kolmogorov-Sinai (KS) entropy. This entropy expression affords a rigorous way of defining the earlier introduced concept of *true entropic index*

Q . If it exists, Q is the value of the entropic index q making the entropy of Eq.(41) increase linearly in time.

To point out the importance of the work of Ref. [24], we have to mention the difficulties with the numerical calculation of Ref. [22]. These authors tried to evaluate the dependence on N of the entropy of Eq.(41) to establish Q in the case of a text of only two symbols, with strong correlations. They have been forced to maintain their numerical study within the range of windows of size $N = 10$. The case of many more symbols would be beyond the range of the current generation of computers. However, when the sequence of symbols is generated by dynamics, as in the case here under study, and the function $\xi(t, x)$ of Eq. (2) is available (for convenience, we keep now explicit the dependence on the initial condition x), it is possible to adopt the prescription of Ref. [24], which writes $H_q(t)$ of Eq.(41) as

$$H_q(t) \equiv \frac{1 - \delta^{q-1} \int dx p(x)^q \xi(t, x)^{1-q}}{q - 1}, \quad (42)$$

where the symbol t denotes time regarded as a continuous variable. In fact, when the condition $N \gg 1$ applies, it is legitimate to identify N with t . The function $p(x)$ denotes the equilibrium distribution density and δ the size of the partition cells: According to Ref. [24] the phase space, a one-dimensional interval, has been divided into $M = 1/\delta$ cells of equal size, with $M \gg 1$.

If the property of Eq.(30) held true with no dependence on the initial condition, namely in the form

$$\xi(t) = [1 - (z - 1)\lambda_z t]^{-z/(z-1)}, \quad (43)$$

one would reach the conclusion that the entropy of Eq.(42) increase linearly in time if we adopt the entropic index of Eq.(21). At the same time the theoretical proposal of Eq.(6) would turn out to be correct. This is the reason why we call this special value of q “magic” and we denote it with the capital letter Q thus writing

$$Q = 1 + (z - 1)/z. \quad (44)$$

What about the case under study, where the local Lyapunov coefficient depends on x ? A possible way out would be given by the following procedure. We assign to the entropic index q of Eq.(42) the value prescribed by Eq.(44). This has the nice effect of making the entropy $H_q(t)$ linear dependent on time. Unfortunately the distribution function $p(x)$ raised to the power of $1 + (z - 1)/z$, results in a divergent contribution as it can be easily checked noticing [21] that $p(x) \propto 1/x^{z-1}$. We notice that it would have been very appealing to establish the “thermodynamic” foundation of $\psi(t)$ on the relation

$$\psi(t) = \langle (\xi(-t))^{1-Q} \rangle_Q^{1/(1-Q)}, \quad (45)$$

with the mean value $\langle A(x) \rangle_Q$ defined by

$$\langle A(x) \rangle_Q \equiv \int dx p(x)^Q A(x). \quad (46)$$

This possibility is ruled out by the divergency stemming from $p(x)^Q$.

We have to point out another important property suggested by Eq.(42). We see that this formula shows that the only possible way of making $H_q(t)$ independent of the cell size δ is to set $q = 1$. In Ref. [28] it has been pointed out that in the multifractal case it is convenient to carry out the average over the distribution of power indices rather than on $p(x)$. This has the nice effect of making $H_q(t)$ independent of the cell size. In the specific case where the invariant distribution exists and is not multifractal, as in the case here under study, we are compelled to use $q = 1$. This is line with the fact that when the invariant distribution of the Manneville map exists, namely if $z < 2$, the average Lyapunov coefficient is finite, the Pesin theorem work and the ordinary KS entropy can be defined [21].

Therefore we are left with the possibility of applying the prescription of Eq.(42), with the associated average of Eq.(46), to the case where $p(x)$ is not the invariant measure, but it is a given out of equilibrium distribution. This would correspond to giving up the assumption that the Tsallis non-extensive thermodynamics has an equilibrium significance, in line with the numerical results of Ref. [29].

On the other hand, it is evident, on the basis of the conclusions of section III, that the function $\xi(t, x)$ does not keep its power-law nature forever, and that in the time scale $t \gg T_{av}$, with T_{av} given by Eq.(19), it is expected to recover an exponential form. Thus, are back to the point that this non-extensive form of thermodynamics must be temporary.

We would be tempted to conclude this section with the following answer to question (iii) of Section I. The Maxwell's Demon thermodynamics is the non-extensive Tsallis thermodynamics, and this is a form of out of equilibrium thermodynamics, implying that also the Maxwell's Demon effect is temporary.

This conclusion, to a first sight, seems to be reasonable. However, it yields a conflict with the result of the earlier work of Ref. [22]. The authors of Ref. [22] studied the entropy of Eq.(41) with a symbolic method, and, as earlier said, they have been forced to use short-time windows. The adoption of short-time windows is imposed by the fact they did not make any direct recourse to dynamics as in the calculation yielding Eq.(43). The case studied in Ref. [22] refers to a dynamic system with the same complexity as the Manneville map here under study. Probably the main difference with the analysis here illustrated is that the method adopted by the authors of Ref. [22] is equivalent to studying the Geisel map rather than the Manneville map, with a repartition of the space into only two cells. The conflict between the analysis here made

and the result of the work of Ref. [22] is probably due to the fact that here we adopt a much finer repartition of the phase space.

This kind of disagreement between two different theoretical predictions about the entropic index of the same dynamic process is not yet a reason for the rebuttal of the non-extensive thermodynamics as the explanation of the Maxwell's Demon effect. In Section V we shall see that the Maxwell's Demon effect seems to be caused by the emergence in chamber A , made more densely populated by the Demon, of a diffusion equation with a "diffusion coefficient" weaker than that of the diffusion equation emerging in chamber B . This dynamic effect is explained using the dynamic derivation of Lévy statistics, which implies memory erasure and consequently emerge in a time scale where the temporary non-extensive thermodynamic regime, if it ever exists, is over.

V. LÉVY PROCESSES AND MEMORY ERASURE

Lebowitz [1] states that the Universe time evolution is characterized by the transition from a condition of lower to a condition of larger entropy without implying a departure from microscopic reversibility. Let us focus our attention on the diffusion process:

$$\frac{dx}{dt} = v(t). \quad (47)$$

Let us assume that the stochastic velocity $v(t)$ fluctuates among the two values W and $-W$, and that the distribution of times of sojourn in one of these two velocity states, the same for the two states, is given by Eq.(18). It is shown [30] that the distribution density $p(x, t)$ obeys the following equation of motion

$$\frac{\partial}{\partial t} p(x, t) = \langle v^2 \rangle \int_0^t \Phi_v(t') \frac{\partial^2}{\partial x^2} p(x, t - t') dt', \quad (48)$$

where $\langle v^2 \rangle$ is the equilibrium mean quadratic value of v , and $\Phi_v(t)$ is the normalized correlation function of the variable v . For details, and especially to understand why this equation in the long-time limit yields the process of Lévy diffusion, the interested reader should consult Ref. [30].

Here we limit ourselves to noticing a few aspects. First of all, the distribution density $p(x, t)$ can be interpreted as a probability concerning a very large number of particles imbedded in the same space phase, for instance, the phase space of the billiards of Zaslavsky [3]. Thus, in a sense, the condition $N \rightarrow \infty$ is fulfilled. This has to do with question (i) of Section I. If condition $N \rightarrow \infty$ is essential for the birth of statistical mechanics, what is the role of mixing, and that of ensuing ergodicity?

We try to answer this question by making a second, very important, remark. This is that Eq.(48) is exact under the following conditions:

(a) The variable $v(t)$ is dichotomous, with only two possible values, either W or $-W$.

(b) All the particles are initially located in the same position, $x = 0$.

We note that condition (a) is that mentioned in Section IV A to make easier for the reader to understand the meaning of the key relation of Eq.(37), leading to the entropic derivation of the waiting time distribution $\psi(t)$. Both conditions are relevant for our main purpose of comparing the point of view of Zaslavsky [3] to that of Lebowitz [1]. The latter property certainly realizes an initial condition whose probability is extremely low, thereby making impossible the regression to the initial condition. In the ideal case of a time inversion, this would be possible, in the unrealistic case of no perturbation of any kind, ranging from the round-off errors to the environmental fluctuations. This is again in line with the observations of Lebowitz on the birth of statistical mechanics.

The former property, however, serves in our opinion a purpose which apparently conflicts with Lebowitz, namely, that of stressing the importance of mixing, as a key property leading to an invariant distribution. For the reader to appreciate this aspect it is necessary that he/she goes through the derivation of Eq.(48) [19]. The projection method adopted in Ref. [19] to derive Eq.(48) rests on a projection operator, which, in turns, implies the existence of an invariant distribution for the variable v . In the specific case of deterministic chaos with mixing, the prototype of which is given in fact by the Bernoulli shift map (as we have shown, the Manneville map with $z = 1$), an invariant distribution exists and is quickly reached from any off-equilibrium condition.

The waiting function $\psi(t)$ is proportional to the second derivative of the correlation function $\Phi_v(t)$. This means that the departure from the exponential condition of Eq.(1) has dramatic effects on the process of transition to statistical mechanics, as birth of a diffusion process. In fact, in the region $1.5 \leq z < 2$ it is still possible to fulfill the condition necessary for the existence of the invariant distribution according to the Kac theorem [9]. However, in this specific case the correlation function Φ_v satisfies the asymptotic property

$$\lim_{t \rightarrow \infty} \Phi_v(t) = \text{const}/t^\beta, \quad (49)$$

with

$$\beta = \frac{z}{z-1} - 2. \quad (50)$$

This means that in the interval $1.5 \leq z < 2$ the index β runs between $\beta = 1$, at the border with the basin of Gaussian attraction, and $\beta = 0$, corresponding to the critical condition of no relaxation of the velocity correlation function.

In conclusion the condition of strong chaos, with strongly mixing properties, has the effect of producing

ordinary Brownian diffusion. It is not clear to us what would be the consequence of a dynamic condition with no form of mixing. We think that in principle it is possible to produce a fast decay of the correlation function $\Phi_v(t)$ as a result of a mere superposition of infinitely many normal modes (again in action the condition $N \rightarrow \infty$). However, we are not aware of any treatment proving that a natural equilibrium distribution is reached also in that case, if no arbitrary statistical assumptions are made.

We have the impression that in the case of ordinary statistics it might be difficult to support the dynamic view against the $(N \rightarrow \infty)$ -perspective, or the latter against the former, without using very subtle arguments such as the request of neither explicit nor implicit statistical assumptions, which, in turn, quite probably would be questioned. For all practical purposes, the two views are expected to provide equivalent results. For instance, in the dynamic perspective adopted by Zaslavsky the Poincaré recurrences are frequent and those with very short time duration are more probable than those of very large time duration (see Eq.(1)). This seems to conflict with Boltzmann's idea that the Poincaré recurrences of systems with very large number of freedoms become exceedingly large, so as to exceed the possibility of any direct observation. However, if we make the assumption that the experimental observation is made on a large number of non-interacting particles, all of them moving in one of the billiards studied by Zaslavsky [3,8,7], we reach a different conclusion. The phase space under study is not more than that of the two-dimensional billiard: It becomes a $2N$ -dimension phase space. The distribution density driven by Eq.(48) becomes a $2N$ -dimensional trajectory, whose return to the initial condition takes place, in accordance with Boltzmann's view, virtually after an infinitely long time. In Zaslavsky's picture the process corresponding to the $2N$ -dimension trajectory leaving the initial condition and returning to it after an extremely long time would correspond to the following scenario. Let us imagine, for instance, that the initial distribution density is different from zero only in a small fraction of the whole 2-dimension phase space, where it is assumed to be constant. As a result of mixing, this initial distribution, throughout its time evolution, would undergo a fragmentation process that, within a kind of coarse-grained perspective, would correspond to the volume of this initial condition growing till to becoming identical to the volume of the whole phase space. The Poincaré recurrences of the Boltzmann-Lebowitz perspective would correspond to this dilated distribution shrinking to the initial condition, a process that is beyond the range of the experimental observation as the return of the Boltzmann-Lebowitz trajectory.

We think that the relevance of dynamics for statistical mechanics cannot be easily ruled out in the case of processes with long-time memory as those here under discussion. Quite on the contrary, here we show that the dy-

dynamic approach to Lévy diffusion in the form discussed in Ref. [30] can account for the Maxwell's Demon effect. In this moment, it is not clear to us how this interesting effect might be explained using the $(N \rightarrow \infty)$ -perspective. The authors of Ref. [30] prove that in a time scale much larger than T_{av} Eq.(48) as an effect of the memory erasure provoked by sporadic randomness becomes equivalent to an equation of motion that in the Fourier representation reads

$$\frac{\partial}{\partial t} \hat{p}(k, t) = -b|k|^{\beta+1} \hat{p}(k, t), \quad (51)$$

with

$$b \equiv \Gamma(-\beta) W^{\beta+1} T_{av}^{\beta} \beta^{(\beta+1)} \cos\left(\frac{\beta}{2}\pi\right). \quad (52)$$

Note that $\hat{p}(k, t)$ is the Fourier transform of $p(x, t)$ and that this is the Fourier representation of the well known process of Lévy diffusion. The element of interest of the result of the dynamic approach of Ref. [30] is that the coefficient b of Eq.(52) plays the role of a diffusion coefficient. The reader can easily realize that the intensity of this coefficient can be different in the two chambers of the experiment on the Maxwell's Demon effect [3,8,7] even if the velocity W is the same. The possibility of realizing two different values for b using the same kinetic energy becomes still wider if we refer to a generator of fluctuations with an inverse power law distribution of waiting times more general than the Manneville map [23]. It is evident that two different diffusion equations in the two chambers can produce a breakdown of the condition of equal population. The trajectory, in fact, will tend to spend more time in the regions with a smaller "diffusion coefficient".

In conclusion, according to this heuristic interpretation, the Maxwell's Demon Effect is permanent and it can show up after the long time process of transition from dynamics to diffusion. This rules out the non-extensive thermodynamics of Tsallis as a possible theoretical interpretation of this effect. In fact, in Section IV B we have seen that if the non-extensive approach applies, it does as a form of temporary thermodynamics.

We can also rule out the non-extensive nature of entropy as the basic property behind the Maxwell's Demon Effect. Let us see why. We have two chambers, A and B , separated by a wall, with a little hole. A can be a Sinai billiard and B a Cassini billiard [7]. The physical entropy of this regime, S , expressed in terms of the conventional Gibbs prescription, might not be extensive. The entropic indicator advocated by Tsallis [12] is not additive, and it violates the property

$$S(A + B) = S(A) + S(B) \quad (53)$$

even when the two subsystems A and B are totally uncorrelated the one from the other. In the case of the

dynamic approach to Lévy diffusion the breakdown of the additivity condition might be dictated by memory effects. To explain why it is so with intuitive arguments, we can refer ourselves to the dynamical experiment itself used by Zaslavsky and Edelman [7] to reveal the existence of Maxwell's Demon effect. A particle that moves from A into B might still have memory of its initial condition in A , if sporadic randomness did not act for a sufficiently extended time. This means that the transition process that in the long-time scale will produce Lévy statistics is not extensive in nature. However, we expect that when Lévy statistics are finally established, the usual additive condition is restored, as a consequence of the memory erasure process. Using the earlier arguments we can imagine the possibility that the two chambers might correspond to two different diffusion equations, even if the velocity intensity of the particle does not change with moving from the one to the other chamber, thereby resulting in the Maxwell's Demon effect after the process of memory erasure.

VI. CONCLUDING REMARKS

Let us summarize our conclusion about the questions raised in Section I.

Question (i). Our arguments concerning question (i) are not compelling and rigorous. However, we hope that they might make plausible the following statement: Both deterministic chaos, whose role is well understood only in the case of low-dimension systems, and the condition $N \rightarrow \infty$ give rise to the birth of statistical mechanics. We are convinced that the joint use of these two perspectives will contribute a deeper understanding of the dynamic origin of statistical mechanics. It is interesting to notice that the exponential of Eq.(1), implying a frequent return to the initial condition, does not conflict with the irreversible character of thermodynamics and statistical mechanics if we perceive the Gibbs ensemble method as being equivalent to the $(N \rightarrow \infty)$ -perspective.

Question(ii). We found the proposal of Eq.(7), valid at the exit time of a trajectory from the laminar region, to be the correct connection between the sensitivity to initial conditions and the distribution of waiting times in the laminar region. This means that in the case of the dynamical systems studied by Zaslavsky, the distribution of the Poincaré recurrence times can be related to the KS entropy only in the absence of intermittency. When an intermittent process is present, the long-time form of the distribution $P_R(t)$ changes from the exponential form of Eq.(1) to an inverse power law form. In the region $1.5 \leq z \leq 2$, corresponding to the emergence of Lévy processes, the KS entropy is still finite and the adoption of Eq.(1) would conflict with the observed inverse power law nature of this distribution. The non-extensive thermodynamics of Tsallis [12] might help, provided that it is viewed as

temporary and provided that the conflict between the prediction about whether $Q > 1$ or $Q < 1$ is settled.

Question (iii). The dynamic approach to Lévy diffusion of Ref. [30] proves that the Maxwell's Demon effect is permanent and it takes place in a long-time scale. This means that it is not necessary to use memory properties to explain it, and consequently, not even the non-extensive nature of entropy.

Question (iv). The condition of ordinary statistical mechanics does not imply an abrupt transition when we set $z = 1$. However, for even infinitesimally small deviations from $z = 1$ in the long time limit the waiting time distribution $\psi(t)$ appears to be characterized by an inverse power law form. This does not have strong consequences, though, because in the region $z < 1.5$ the second moment of the waiting time distribution is finite, and consequently the system is attracted by the Gauss basin of attraction. Nevertheless, the waiting time distribution in the region where z is very close to 1, is an exponential at short time and an inverse power law at long time, a property which makes it incompatible with the structure generated by the method of entropy maximization resting on the non-extensive indicator of Tsallis [12].

It seems to us that the long-standing problem of a thermodynamic approach to Lévy processes [31] is not yet solved, and that the Maxwell's Demon effect of Refs. [8,3,7] can be regarded as a paradigmatic case challenging the Boltzmann perspective advocated by Lebowitz [1]. We plan to do further research work along these directions.

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